Analysis IV

(for EL, GM, MX)

Week 9 - 14.04.25

Recall from last time:

Laplace transform: If $f: [0,+\infty) \longrightarrow \mathbb{C}$ is piecewise continuous, then

$$\mathcal{L}[f](z) = \int_0^{+\infty} f(t)\cdot e^{-tz}\,dt.$$

Whenever the integral converges.

- Morally speaking: The integral converges if, as $t \to +\infty$, $|f(t)|\cdot e^{-\mathrm{Re}(z)\cdot t} \longrightarrow 0$ at a rate which is "integrable".

Abscissa of convergence. $\gamma_0 \in \mathbb{R}$ is called an abscissa of convergence for $\mathcal{L}[f]$ if, for any $\gamma > \gamma_0$, we have

$$\int_0^{+\infty} |f(t)|\cdot e^{-\gamma t}\,dt < +\infty.$$

So if γ_0 is an abscissa of convergence for $\mathcal{L}[f]$: $\mathcal{L}[f](z)$ is holomorphic in the half-plane

$$U = \{z \in \mathbb{C} : \operatorname{Re}(z) > \gamma_0\}.$$

- Sufficient condition: If $\int_0^{+\infty} |f(t)|\, e^{-\gamma_0 t}\, dt < +\infty$, then γ_0 is an abscissa of convergence (since the same integral for $\gamma > \gamma_0$ then necessarily is bounded)
- Lemma from the exercises last week:

If γ_0 is an abscissa of convergence for $\mathcal{L}[f(t)]$, then it is also an abscissa of convergence for $\mathcal{L}[t^n f(t)]$, $n \in \mathbb{N}$.

- We also proved last week:

$$\boxed{\frac{d}{dz}\mathcal{L}[f](z) = -\mathcal{L}[t \cdot f]} \quad (*)$$

- We also showed that, if $f(t) = 1$ (constant) then 0 is an abscissa of convergence for $\mathcal{L}[f]$ (since, for any $\gamma > 0$, $\int_0^{+\infty} 1 \cdot e^{-\gamma t} dt < +\infty$) and

$$\boxed{\mathcal{L}[1](z) = \frac{1}{z}}$$

(Since 0 is an abscissa of convergence, the above relation holds for $Re(z) > 0$).

- Using $(*)$ and differentiating the above relation, we have for any z with $Re(z) > 0$ and any $n \in \mathbb{N}$:

$$\boxed{\mathcal{L}[t^n](z) = \left(-\frac{d}{dz}\right)^n \left(\frac{1}{z}\right) = \frac{n!}{z^{n+1}}}$$

More properties of the Laplace transform:

Linearity: Let $f, g : [0, +\infty) \longrightarrow \mathbb{C}$ be such that $\mathcal{L}[f]$, $\mathcal{L}[g]$ are defined for $\mathrm{Re}(z) > \gamma_0$, then, for any constants $a, b \in \mathbb{C}$,

$$\mathcal{L}[af + bg](z) = a\,\mathcal{L}[f](z) + b\,\mathcal{L}[g](z)$$

is defined for $\mathrm{Re}(z) > \gamma_0$.

Theorem Let $f : [0, +\infty) \longrightarrow \mathbb{C}$ be of class C^1 (that is to say, f' is continuous). Assume that γ_0 is an abscissa of convergence for $\mathcal{L}[f]$ and $\mathcal{L}[f']$. Then, for $\mathrm{Re}(z) > \gamma_0$:

$$\boxed{\mathcal{L}[f'](z) = z \cdot \mathcal{L}[f](z) - f(0)}$$

Remark: Similarly as in the case of the Fourier transform, the Laplace transform transforms derivatives into multiplication with z (so differential equations are transformed to algebraic equations). A crucial difference is that the Laplace transform of f' also picks up the "initial value" $f(0)$.

Proof of the theorem:

$$\mathcal{L}[f'](z) = \int_0^{+\infty} f'(t) \cdot e^{-tz} \, dt.$$

We would like to integrate by parts. We will do this carefully, in order to deal with the boundary term at $t = +\infty$ appropriately:

Since $\mathrm{Re}(z) > \gamma_0$ and γ_0 is an abscissa of convergence for $\mathcal{L}[f']$, $\int_0^{+\infty} f(t) e^{-tz} dt$ converges absolutely. So, for any sequence $T_n \xrightarrow{n \to +\infty} +\infty$ (which we will define precisely later), we have

$$\mathcal{L}[f'](z) = \int_0^{+\infty} f'(t)\, e^{-tz}\, dt$$

$$= \lim_{n \to +\infty} \int_0^{T_n} f'(t)\, e^{-tz}\, dt$$

Integration by parts

$$= \lim_{n \to +\infty} \left(\left[f(t) \cdot e^{-tz} \right]_{t=0}^{t=T_n} - \int_0^{T_n} f(t) \cdot (-z) e^{-tz} dt \right)$$

$$= \lim_{n \to +\infty} \left(f(T_n) e^{-T_n z} \right) - f(0) \cdot e^0$$

$$+ z \cdot \int_0^{+\infty} f(t)\, e^{-tz}\, dt$$

$$= \lim_{n\to+\infty} f(T_n)\cdot e^{-T_n\cdot z} - f(0) + z\cdot \mathcal{L}[f](z).$$

We will now show that we can choose the sequence $T_n \longrightarrow +\infty$ so that $\lim_{n\to+\infty} f(T_n)\cdot e^{-T_n z} = 0$

(this will give the formula that we want).

Recall that γ_0 is an abscissa of convergence for $\mathcal{L}[F]$ and $Re(z) > \gamma_0$, so that

$$\int_0^{+\infty} |f(t)|\cdot|e^{-z\cdot t}|\,dt = \int_0^{+\infty} |f(t)|\cdot e^{-Re(z)\cdot t}\,dt < +\infty.$$

Therefore, there must exist a sequence $T_n \to +\infty$ such that $|f(T_n)|\cdot|e^{-z\cdot T_n}| \xrightarrow{n\to+\infty} 0$

(otherwise, the function $|f(t)|\cdot|e^{-zt}|$ must stay bounded away from 0 as $t \to +\infty$, which would contradict the bound $\int_0^{+\infty} |f(t)| \cdot |e^{-zt}| dt < +\infty$).

For this sequence T_n: $\lim\limits_{n \to +\infty} f(T_n) \cdot e^{-zT_n} \longrightarrow 0$.

$\blacksquare$

More generally, iterating the above formula:

If $f \in C^n([0,+\infty);\mathbb{C})$ (this notation means that $f : [0,+\infty) \longrightarrow \mathbb{C}$ is n-times differentiable and $f, f', \dots, f^{(n)}$ are continuous), we have:

$$\mathcal{L}[f'](z) = z \cdot \mathcal{L}[f](z) - f(0)$$

$$\mathcal{L}[f''](z) = z \cdot \mathcal{L}[f'](z) - f'(0)$$

$$= z^2 \cdot \mathcal{L}[f](z) - z \cdot f(0) - f'(0)$$

$$\vdots$$

$$\boxed{\begin{aligned} \mathcal{L}[f^{(n)}](z) &= z^n \cdot \mathcal{L}[f](z) \\ &\quad - z^{n-1} \cdot f(0) - \dots - z \cdot f^{(n-2)}(0) - f^{(n-1)}(0) \\ (**) \qquad &= z^n \cdot \mathcal{L}[f](z) - \sum_{j=0}^{n-1} z^{n-1-j} f^{(j)}(0). \end{aligned}}$$

Example: Consider $f(t) = t^2$, $f'(t) = 2t$, $f''(t) = 2$. Their Laplace transform:

Defined for $\mathrm{Re}(z) > 0$ (we already calculated them, but we can rederive the formulas using $(**)$)

$$f(t) = t^2$$

$$f'(t) = 2t$$

$$f''(t) = 2$$

$$\begin{tikzcd} \mathcal{L}[f] = 2/z^3 \arrow[d, bend right, "\times z"'] \\ \mathcal{L}[f'] = 2/z^2 \arrow[u, bend right, "\times \frac{1}{z}"'] \arrow[d, bend right, "\times z"'] \\ \mathcal{L}[f''] = 2/z \arrow[u, bend right, "\times \frac{1}{z}"'] \end{tikzcd}$$

Theorem: Let $f:[0,+\infty)\to\mathbb{C}$ be piecewise continuous and $\gamma_0 \geq 0$ an abscissa of convergence for $\mathcal{L}[f]$.

$$\boxed{\text{If } g(t)=\int_0^t f(s)\,ds \Rightarrow \mathcal{L}[g](z)=\frac{1}{z}\cdot\mathcal{L}[f](z) \quad \text{for } \mathrm{Re}(z)>\gamma_0}$$

Remark: In general, the abscissa of convergence for g cannot be negative (even if it is for f).

Sketch of pf: If $g(t)=\int_0^t f(s)\,ds$, then $f(t)=g'(t)$ and $g(0)=0$.

So:
$$\mathcal{L}[g'](z) = z\cdot\mathcal{L}[g](z)-\cancel{g(0)}^{\,0}$$
$$= z\cdot\mathcal{L}[g](z)$$

$$\Longrightarrow \quad \mathcal{L}[g](z)=\frac{1}{z}\cdot\mathcal{L}[g'](z)=\frac{1}{z}\cdot\mathcal{L}[f](z).$$

(we will not show why $\mathcal{L}[g](z)$ is indeed defined for $Re(z) > \gamma_0$ when $\gamma_0 \geq 0$). ∎

Theorem: Let $a > 0$ and $b \in \mathbb{R}$. Let $f : [0, +\infty) \to \mathbb{C}$ be piecewise continuous with abscissa of convergence $\gamma_0 \in \mathbb{R}$. Let

$$g(t) = e^{-bt} \cdot f(at)$$

Then

$$\boxed{\mathcal{L}[g](z) = \frac{1}{a} \mathcal{L}[f]\left(\frac{z+b}{a}\right)}$$

for any $z \in \mathbb{C}$ with $Re\left(\frac{z+b}{a}\right) > \gamma_0$

$$\overset{a>0,\, b\in\mathbb{R}}{\iff} \quad Re(z) > a\gamma_0 - b$$

(and, in particular, $a\gamma_0 - b$ is an abscissa of convergence for $\mathcal{L}[g]$)

Sketch of the proof:

We will not show that $a\gamma_0 - b$ is an abscissa of convergence (Exercise!), but we will compute the formula for $\mathcal{L}[g]$:

$$\mathcal{L}[g](z) = \int_0^{+\infty} e^{-bt} f(a\cdot t)\, e^{-zt}\, dt$$

$$= \int_0^{+\infty} f(at) \cdot e^{-(z+b)t}\, dt$$

$s = at$ $$= \int_0^{+\infty} f(s) \cdot e^{-(z+b)\cdot\frac{s}{a}} \cdot \frac{ds}{a}$$

$$= \frac{1}{a} \cdot \int_0^{+\infty} f(s) \cdot e^{-\left(\frac{z+b}{a}\right)s}\, ds$$

$$= \frac{1}{a} \mathcal{L}[f]\left(\frac{z+b}{a}\right).$$

$\boxed{}$

Remark: Compare the above formula with the Formula for the Fourier transform of the rescaling $f(t) \longrightarrow f(\lambda \cdot t)$.

Example: $g(t) = e^{ct} = e^{ct} \cdot 1$, .

Here $a = 1$, $b = -c$. According to the theorem:

$$\boxed{\mathcal{L}[g](z) = \mathcal{L}[1](z-c) = \frac{1}{z-c}}$$

for $\operatorname{Re}(z) > c$.

Rmk: In this case: We can compute that $\gamma_0 = c$ is an abscissa of convergence for $\mathcal{L}[e^{ct}]$ without using the theorem.

For any $\gamma > c$, we have

$$\int_0^{+\infty} |e^{ct}| \cdot |e^{-\gamma t}| \, dt = \int_0^{+\infty} e^{ct} \cdot e^{-\gamma t} \, dt$$

$$= \int_0^{+\infty} e^{-(-c+\gamma)t} \, dt$$

$$= \left[\frac{e^{-(\gamma - c)t}}{-(\gamma - c)} \right]_{t=0}^{t=+\infty} = \frac{1}{\gamma - c} < +\infty$$

since $\gamma > c \iff \gamma - c > 0$

(so $e^{-(\gamma - c)t} \xrightarrow{t \to +\infty} 0$).

Example: Find an $f : [0, +\infty) \to \mathbb{C}$ such that $\mathcal{L}[f](z) = \dfrac{1}{(z-a)(z-b)}$, $a, b \in \mathbb{R}$.

Notice: $\mathcal{L}[f](z) = \dfrac{1}{(z-a)(z-b)} =$

$$= \frac{1}{b-a}\left(\frac{1}{z-b} - \frac{1}{z-a}\right).$$

$$= \frac{1}{b-a}\cdot\left(\mathcal{L}[e^{bt}](z) - \mathcal{L}[e^{at}](z)\right)$$

$$= \frac{1}{b-a}\cdot\mathcal{L}\left[e^{bt} - e^{at}\right](z)$$

$$= \mathcal{L}\left[\frac{e^{bt} - e^{at}}{b-a}\right](z).$$

Defined for $\mathrm{Re}(z) > \max\{a, b\}$.

Convolutions

Let $f, g : [0, +\infty) \longrightarrow \mathbb{C}$ be piecewise continuous functions.

We extend f, g on all of $\mathbb{R}$ by assuming they are $\equiv 0$ on $(-\infty, 0)$.

$$f * g(t) = \int_{-\infty}^{+\infty} f(t-s)\, g(s)\, ds$$

$f(t-s) = 0$ for $t-s \leq 0$ and $g(s) = 0$ for $s \leq 0$

$$= \begin{cases} \int_0^t f(t-s)\, g(s)\, ds & \text{if } t \geq 0 \\ 0, & \text{if } t \leq 0. \end{cases}$$

So: $f * g$ can be restricted back on $[0, +\infty)$ (its values on $(-\infty, 0)$ are trivial).

In this way: $f * g$ is well defined as an operation for functions on $[0, +\infty)$.

Theorem

Let $f, g : [0, +\infty) \longrightarrow \mathbb{C}$ be piecewise continuous functions with $\gamma_0 \in \mathbb{R}$ an abscissa of convergence for $\mathcal{L}[f], \mathcal{L}[g]$.

Then

$$\boxed{\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]}$$

has abscissa of convergence γ_0.